Strong Nœther Position[*]

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Abstract

In this paper, we introduce the notion of a homogeneous ideal in *strong Nœther position* (SNP); a new definition for the notion of generic coordinates for some problems. This definition is simple to check, because one can test it for the initial ideal of the ideal with respect to the degree reverse lexicographic ordering. It is explicit, because we can provide an algorithm to decide whether a monomial ideal is in SNP or not. We propose some methods to compute the Castelnuovo-Mumford regularity of an ideal which one of them is more efficient than that of [BG05]. Also, to any notion of regularity of an ideal (Castelnuovo-Mumford regularity, satiety, maximal degree of the Gröbner basis elements and Hilbert regularity), we associate a *stabilized regularity.* The main result of this paper is that all of these regularities are equal in characteristic zero if the ideal is in SNP.

1 Introduction

The aim of this paper is to present the notion of a homogeneous ideal in *strong Nœther position* (SNP). This is a new explicit definition for the concept *generic coordinates*, which allows to compute efficiently the regularities of an ideal from its initial ideal with respect to the degree reverse lexicographic ordering. We show that for almost all linear change of variables, an ideal is in SNP. SNP is easily tested on initial ideal. This and the equivalence property of the definition of SNP allows to improve [BG05] algorithm for computing the Castelnuovo-Mumford regularity of an ideal. For any regularity of an ideal, we associate a *stabilized regularity.* It is defined by the maximal of the corresponding regularities of the ideals obtained by adding the last variables to the given ideal.

We prove that if an ideal is in SNP, its stabilized Castelnuovo-Mumford regularity, stabilized satiety and stabilized Hilbert regularity are all equal to the Castelnuovo-Mumford regularity, and they are *an upper bound for the maximal degree of the elements of the reduced Gröbner basis of the ideal with respect to the degree reverse lexicographic ordering.* Moreover, this inequality becomes an equality if the characteristic of the base field is zero.

To explain the motivation of introducing SNP, let I be a homogeneous ideal of a polynomial ring R. A good measure to estimate the complexity of the computation of the Gröbner basis of I is the maximal degree of the polynomials which appear in this computation (see [Laz81, Laz83, Giu84]). By a well-known result, the Castelnuovo-Mumford regularity of I is an upper bound for this degree in generic coordinates and for the degree reverse lexicographic ordering. This upper bound is reached if the characteristic of K is zero (see [Ang84, BS87]). In fact, SNP is an explicit definition of generic coordinates for this problem. Bayer and Stillman in [BS87] have already introduced an explicit definition of generic coordinates. On the other hand, Bermejo and Gimenez in [BG05] have defined the notion of nested type for a monomial ideal. SNP may be considered as a generalization of both definitions for any ideal.

2 Definition of strong Nœther position

In this section, we define the concept of a homogeneous ideal in strong Nœther position (SNP). We give three definitions (Definitions 2.1, 2.2 and 2.3), and we prove that they are equivalent. This equivalence

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defines SNP. For this, let I be a homogeneous ideal of the ring $R = K[x_0, \dots, x_n]$ where K is an arbitrary field. Let $R_i = K[x_0, \dots, x_i]$. We need the following notations.

Notation 2.1 *Let* $\sec(I,0) = \overline{\sec}(I,0) = \widehat{\sec}(I,0) = I$ *and for* $i = 1, \dots, n+1$ *let*

- $\sec(I,i) = I + \langle x_{n-i+1}, \dots, x_n \rangle$;
- $\overline{\sec}(I,i) = \sec(I,i) \cap R_{n-i} = I|_{x_{n-i+1}=\cdots=x_n=0} \cap R_{n-i}$;
- $\widehat{\sec}(I,i) = \big(\widehat{\sec}(I,i-1) : x_{n-i+1}^{\infty}|_{x_{n-i+1}=0}\big) \cap R_{n-i}$.

The *saturation* of I, denoted by I^{sat}, is the maximal ideal $J \subset R$ for the following property: The homogeneous part I_m and J_m of degree m of I and J are equal for all m large enough.

Definition 2.1 *The ideal I is in* SNP1 *if* $\sec(I,i)^{\text{sat}} = \sec(I,i) : x_{n-i}^{\infty}$ *for* $i = 0, \dots, \dim(I) - 1$.

Definition 2.2 *The ideal I is in* SNP2 *if* $\overline{\sec}(I,i)^{\text{sat}} = \overline{\sec}(I,i) : x_{n-i}^{\infty}$ *for* $i = 0, \dots, \dim(I) - 1$.

Definition 2.3 *The ideal I is in* SNP3 *if* $\widehat{\sec}(I,i)^{\text{sat}} = \widehat{\sec}(I,i) : x_{n-i}^{\infty}$ *for* $i = 0, \dots, \dim(I) - 1$.

We prove the equivalence of these definitions, and this enables us to define SNP.

Definition 2.4 *An ideal I is called in* SNP*, if it is in* SNP1 *or in* SNP2 *or in* SNP3.

Recall that the degree reverse lexicographic monomial ordering, denoted by $\prec$, in R is defined on monomials of the same degree by $x_0^{\alpha_0} \cdots x_n^{\alpha_n} \prec x_0^{\beta_0} \cdots x_n^{\beta_n}$ if the last non-zero entry of the vector $(\beta_0 - \alpha_0, \dots, \beta_n - \alpha_n)$ is negative. If $F \in R$ is a homogeneous polynomial, we denote by $\text{in}(F)$ the greatest monomial (leading term) of F with respect to $\prec$, and $\text{in}(I) = \{\text{in}(F) \mid F \in I\}$.

Theorem 2.1 *Let $I \subset R$ be a homogeneous ideal. Then I is in* SNP *if and only if* $\text{in}(I)$ *is in* SNP.

Since for almost all linear changes of variables a monomial ideal is in SNP1 by [Gal74] (see also [BG05], Theorem 4.4), then a homogeneous ideal is in SNP for almost all linear changes of variables.

We give also the following criterion to check whether a monomial ideal is in SNP or not. Using this test and the above theorem, we may derive an algorithm to decide whether an ideal is in SNP or not.

Proposition 2.1 **(SNP-test)** *Let $J \subset R$ be a monomial ideal. The following conditions are equivalent:*

(1) *J is in* SNP;

(2) $J : x_{n-i}^{\infty} = J : \langle x_0, \dots, x_{n-i} \rangle^{\infty}$ *for each* $i = 0, \dots, \dim(J) - 1$;

(3) *For any $i > n - \dim(J)$ and any generator $m = x_0^{e_0} \cdots x_i^{e_i}$ with $e_i > 0$, $x_{i-1}^{k} m / x_i^{e_i} \in J$ for some k.*

Corollary 2.1 *There is a polynomial algorithm (with respect to the number of generators) to test if a monomial ideal is in* SNP.

3 Computing satiety and Castelnuovo-Mumford regularity

In this section, we propose some methods to compute the satiety and Castelnuovo-Mumford regularity of an ideal in SNP. The *satiety* of I, $\text{sat}(I)$, is the smallest integer m such that $I_\ell = I_\ell^{\text{sat}}$ for all $\ell \geq m$.

Proposition 3.1 *Let $I \subset R$ be a homogeneous ideal. If I is in* SNP *then*

$$\text{sat}(I) = \text{sat}(\text{in}(I)) = \max_{m \in \text{in}(I):x_n \setminus \text{in}(I)} \{\deg(m)\} + 1.$$

Let us define the *Castelnuovo-Mumford regularity* of a homogeneous ideal I. If

$$0 \longrightarrow \bigoplus_j R(e_{rj}) \longrightarrow \cdots \longrightarrow \bigoplus_j R(e_{1j}) \longrightarrow \bigoplus_j R(e_{0j}) \longrightarrow I \longrightarrow 0$$

is a minimal graded free resolution of I, then $\text{reg}(I)$ is the maximal of $e_{ij} - i$ for each i and j.

Theorem 3.1 *Let $I \subset R$ be a homogeneous ideal of dimension d which is in* SNP*. Then*

$$\begin{aligned} \mathrm{reg}(I) = \mathrm{reg}(\mathrm{in}(I)) &= \max_{0 \le i \le d} \{\mathrm{sat}(\mathrm{sec}(\mathrm{in}(I), i))\} & (1) \\ &= \max_{0 \le i \le d} \{\mathrm{sat}(\overline{\mathrm{sec}}(\mathrm{in}(I), i))\} & (2) \\ &= \max_{0 \le i \le d} \{\mathrm{sat}(\widehat{\mathrm{sec}}(\mathrm{in}(I), i))\}. & (3) \end{aligned}$$

These equalities remain true if $\mathrm{in}(I)$ *is replaced by I in the right members of these equalities.*

Remark 3.1 Formula (3) is an efficient tool to compute the Castelnuovo-Mumford regularity of a homogeneous ideal I in SNP. Since $\widehat{\mathrm{sec}}(\mathrm{in}(I), i) \subset R_{n-i}$ is in SNP for any i then its satiety is equal to

$$\max_{m \in \widehat{\mathrm{sec}}(\mathrm{in}(I), i) : x_{n-i} \setminus \widehat{\mathrm{sec}}(\mathrm{in}(I), i)} \{\deg(m)\} + 1$$

by Proposition 3.1. Bermejo and Gimenez [BG05] (see Theorem 3.7) compute the satiety with a similar formula where $\widehat{\mathrm{sec}}(\mathrm{in}(I), i) : x_{n-i}$ is replaced by $\widehat{\mathrm{sec}}(\mathrm{in}(I), i) : \mathfrak{m}_i$ where $\mathfrak{m}_i$ is the unique maximal ideal of R_{n-i}. As $I : x$ is easier to compute than $I : \mathfrak{m}$, the notion of SNP allows to improve their algorithm.

Example 3.1 *Computing the Castelnuovo-Mumford regularity of an ideal in* SNP *by formula (3).* Let $R = K[x, y, z, t, w]$. Consider the following monomial ideal from [BG05], Example 3.13

$$J = \langle x^4, x^3y, x^2y^2, y^4, x^3z, x^2z^2, y^3z^5, x^3t, x^3w^2 \rangle.$$

This is an ideal of dimension 3, and in SNP (Proposition 2.1(3)). We have $\widehat{\mathrm{sec}}(J, 0) = J$, and $\widehat{\mathrm{sec}}(J, 0) : w = J + \langle x^3w \rangle$ and therefore $\mathrm{sat}(\widehat{\mathrm{sec}}(J, 0)) = \deg(x^3w) + 1 = 5$ by Proposition 3.1. Since $x^3 \in \widehat{\mathrm{sec}}(J, 0) : w^\infty$ then $\widehat{\mathrm{sec}}(J, 1)$ is generated by monomials independent of t, and its satiety is zero. Now, $\widehat{\mathrm{sec}}(J, 2) : z = \widehat{\mathrm{sec}}(J, 2) + \langle x^2z, y^3z^4 \rangle$. So $\mathrm{sat}(\widehat{\mathrm{sec}}(J, 2)) = \deg(xy^3z^4) + 1 = 9$. Finally, $\widehat{\mathrm{sec}}(J, 3) : y = \widehat{\mathrm{sec}}(J, 3) + \langle y^2 \rangle$. Therefore, $\mathrm{sat}(\widehat{\mathrm{sec}}(J, 3)) = \deg(xy^2) + 1 = 4$. Thus, $\mathrm{reg}(J) = \max\{5, 0, 9, 4\} = 9$ by Theorem 3.1. In this example, one can see that $\widehat{\mathrm{sec}}(J, 2) : z$ may be computed easier than $\widehat{\mathrm{sec}}(J, 2) : \langle x, y, z \rangle$. Using the fonction `quotient` of the software `Singular` (see [GPS05]) the first one is equal to $x^2z, x^3, y^4, x^2y^2, y^3z^4$ while the second one is equal to $x^3, x^2z^2, x^2yz, y^4, x^2y^2, y^3z^5, xy^3z^4$.

4 Stabilized regularities

In this section, we associate to each notion of regularity for an ideal (Castelnuovo-Mumford regularity, satiety, degree of the Gröbner bases and Hilbert regularity defined further) a *stabilized regularity*. The principal result of this section is that all these regularities are equal in characteristic zero, if the ideal is in SNP. It is this result which motivated the introduction of the SNP and which makes it interesting.
The *Hilbert function* of a homogeneous ideal $I \subset R = K[x_0, \ldots, x_n]$ is defined by $\mathrm{HF}_I(t) = \dim_K(R/I)_t$ where $\dim_K(R/I)_t$ is the dimension of the set of the elements of degree t of R/I as a K-space vectorial. From a certain degree, this function of t is equal to a polynomial in t; we note this polynomial by HP_I.

Definition 4.1 *The* Hilbert regularity *of I is*

$$\mathrm{hilb}(I) = \min\{m \mid \forall t \ge m,\ \mathrm{HF}_I(t) = \mathrm{HP}_I(t)\}.$$

We need also a definition of the degree of the reduced Gröbner basis of a homogeneous ideal.

Definition 4.2 *Let us note $\deg(I, \prec)$ the maximal degree of the elements of the reduced Gröbner basis of I with respect to $\prec$.*

Now, we define the stabilized regularities.

Definition 4.3 *For each homogeneous ideal $I \subset R$, the* stabilized Hilbert regularity *of I is*

$$\overline{\mathrm{hilb}}(I) = \max\left\{\mathrm{hilb}(\mathrm{sec}(I, i)) \middle|\ 0 \le i \le \dim\left(\frac{R}{I}\right)\right\}.$$

The quantities $\overline{\mathrm{reg}}$, $\overline{\mathrm{sat}}$ et $\overline{\mathrm{deg}}$ are defined in a similar way.

Theorem 4.1 *Let $I \subset R$ be a homogeneous ideal in* SNP. *Then*

$$\deg(I, \prec) = \overline{\deg}(I, \prec) \leq \mathrm{reg}(I) = \overline{\mathrm{reg}}(I) = \overline{\mathrm{hilb}}(I) = \overline{\mathrm{sat}}(I).$$

Moreover, the inequality becomes an equality if the characteristic of K is zero.

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